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Finite Range Effects in Coulomb Stripping

William F. Ford

National Aeronautics and Space Administration

Lewis Research Center, Cleveland, Ohio

When nuclear rearrangement collisions are dominated by Coulomb forces, as in the case of high-Z targets and/or projectiles at low bombarding energies, it is often possible to obtain a good approximation to the cross section which can be expressed in closed form. Recently Morinigo [1] has developed such an expression for L = 0 stripping reactions, under the assumption of zero-range forces. In this note we shall indicate how the theory may be extended to include finite-range effects, at least to first order, valid for all L.

We consider the reaction (AB) + C $\longrightarrow$ A + (BC), for which the transition amplitude is given approximately by

$$T_{if} = \left\langle \phi_f(\underset{\sim}{R}')\, \psi^{(-)}_{\underset{\sim}{k}'}(\underset{\sim}{r}') \middle| V_{AB}(\underset{\sim}{R}) \middle| \phi_i(\underset{\sim}{R})\, \psi^{(+)}_{\underset{\sim}{k}}(\underset{\sim}{r}) \right\rangle . \qquad (1)$$

Here, $\phi_f(\underset{\sim}{R}')$ denotes the final (BC) bound state and $\phi_i(\underset{\sim}{R})$ the initial (AB) bound state, while $\psi^{(-)}_{\underset{\sim}{k}'}(\underset{\sim}{r}')$ and $\psi^{(+)}_{\underset{\sim}{k}}(\underset{\sim}{r})$ denote the scattering states of A relative to (BC) and (AB) relative to C, respectively. The coordinates are given by $\underset{\sim}{R}' = \underset{\sim}{r}_B - \underset{\sim}{r}_C$, $\underset{\sim}{R} = \underset{\sim}{r}_A - \underset{\sim}{r}_B$, $\underset{\sim}{r}' = \underset{\sim}{r}_A - \underset{\sim}{R}_{BC}$, and $\underset{\sim}{r} = \underset{\sim}{R}_{AB} - \underset{\sim}{r}_C$, where $\underset{\sim}{R}_{BC}$ and $\underset{\sim}{R}_{AB}$ are the center-of-mass coordinates of (BC) and (AB).

The derivation of eq. (1) is quite standard; see, for example, ref. [2]. The next assumption, however, that $\psi^{(-)}_{\underset{\sim}{k}'}$ and $\psi^{(+)}_{\underset{\sim}{k}}$ may be regarded as pure Coulomb waves, is the heart of the present theory, and depends for its

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validity on the fact that the Coulomb barrier depresses the wave function in just the regions where the nuclear forces are expected to distort it most noticeably. This point has been discussed at length in refs. [3,4]; here we shall simply assume that all of the necessary conditions have been met.

Now in Morinigo's zero-range theory, V_{AB} is approximated by a δ-function, thus reducing the expression for T_{if} to a single integral which can be evaluated in closed form. Here we treat V_{AB} more accurately, by relating it to the bound state ϕ_i and making use of an important property of Coulomb wave functions.

We begin by introducing the Fourier transforms of $\phi(\underset{\sim}{R})$ and $\psi_{\underset{\sim}{k}}(\underset{\sim}{r})$, which will be denoted by $\phi(\underset{\sim}{P})$ and $\psi_{\underset{\sim}{k}}(\underset{\sim}{p})$. The transition amplitude may then be written

$$T_{if} = \iint \left[\phi_f(\underset{\sim}{P}')\,\psi^{(-)}_{\underset{\sim}{k}'}(\underset{\sim}{p}')\right]^* F_i(\underset{\sim}{P})\,\psi^{(+)}_{\underset{\sim}{k}}(\underset{\sim}{p})\,d\underset{\sim}{p}\,d\underset{\sim}{p}', \qquad (2)$$

where $\underset{\sim}{P} = \underset{\sim}{p}' - \xi\underset{\sim}{p}$ and $\underset{\sim}{P}' = \underset{\sim}{p} - \xi'\underset{\sim}{p}'$, with $\xi = M_A/(M_A + M_B)$ and $\xi' = M_C/(M_C + M_B)$. The function $F_i(\underset{\sim}{P})$ is the Fourier transform of $V_{AB}(\underset{\sim}{R})\,\phi_i(\underset{\sim}{R})$; thus we may write

$$F_i(\underset{\sim}{P}) = -(\epsilon_{AB} + \hbar^2P^2/2M_{AB})\,\phi_i(\underset{\sim}{P}) = -(\hbar^2/2M_{AB})(\alpha^2 + P^2)\,\phi_i(\underset{\sim}{P}), \qquad (3)$$

where M_{AB} is the reduced mass of the system (AB) and ϵ_{AB} is its binding energy.

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Both $F_i(\underset{\sim}{P})$ and $\underset{-}{\phi}_f(\underset{\sim}{P}')$ are fairly smooth functions of their arguments, peaked near the origin; of them $F_i(\underset{\sim}{P})$ is the flatter, and in fact is constant in the zero-range approximation to V_{AB}. In contrast, the scattering transforms are very sharply peaked functions, as may be seen from the equation they satisfy:

$$\underset{-}{\psi}_{\underset{\sim}{k}}^{(\pm)}(\underset{\sim}{p}) = (2\pi)^3 \underset{-}{\delta}(\underset{\sim}{p} - \underset{\sim}{k}) + \frac{1}{E_k + i\underset{-}{\epsilon} - E_p} T_{\underset{\sim}{k}}^{(\pm)}(\underset{\sim}{p}), \tag{4}$$

$$T_{\underset{\sim}{k}}^{(\pm)}(\underset{\sim}{p}) = \int e^{-i\underset{\sim}{p}\cdot\underset{\sim}{r}} V(\underset{\sim}{r})\, \underset{-}{\psi}_{\underset{\sim}{k}}^{(\pm)}(\underset{\sim}{r})\, d\underset{\sim}{r}. \tag{5}$$

For short-range forces the T matrix $T_{\underset{\sim}{k}}(\underset{\sim}{p})$ is relatively slowly varying, but for the long-range Coulomb force $V(\underset{\sim}{r}) = V_o/r$ one finds that [5,6]

$$T_{\underset{\sim}{k}}^{(\pm)}(\underset{\sim}{p}) \propto \frac{1}{(\underset{\sim}{p} - \underset{\sim}{k})^2} \left[\frac{p^2 - k^2}{(\underset{\sim}{p} - \underset{\sim}{k})^2}\right]^{\pm i\underset{-}{\eta}}; \tag{6}$$

thus both terms in eq. (4) are singular at $\underset{\sim}{p} = \underset{\sim}{k}$. Since the integrand of eq. (2) contains the factor $\underset{-}{\psi}_{\underset{\sim}{k}'}^{(-)}(\underset{\sim}{p}')^* \underset{-}{\psi}_{\underset{\sim}{k}}^{(+)}(\underset{\sim}{p})$, the main contribution to the integral comes from a region about the point $\underset{\sim}{p} = \underset{\sim}{k}$, $\underset{\sim}{p}' = \underset{\sim}{k}'$, and we may remove the slowly varying $F_i(\underset{\sim}{P})$ from under the integral sign [+] as follows:

$$T_{if} \cong F_i(\underset{\sim}{k}' - \underset{-}{\zeta}\underset{\sim}{k}) \iint \left[\underset{-}{\phi}_f(\underset{\sim}{P}')\, \underset{-}{\psi}_{\underset{\sim}{k}'}^{(-)}(\underset{\sim}{p}')\right]^* \underset{-}{\psi}_{\underset{\sim}{k}}^{(+)}(\underset{\sim}{p})\, d\underset{\sim}{p}\, d\underset{\sim}{p}'$$

$$= (2\pi)^{\frac{3}{2}} F_i(\underset{\sim}{k}' - \underset{-}{\zeta}\underset{\sim}{k}) \left\langle \underset{-}{\phi}_f(\underset{\sim}{r})\, \underset{-}{\psi}_{\underset{\sim}{k}'}^{(-)}(\underset{-}{\zeta}'\underset{\sim}{r}) \,\middle|\, \underset{-}{\psi}_{\underset{\sim}{k}}^{(+)}(\underset{\sim}{r}) \right\rangle. \tag{7}$$

From this point on we may use Morinigo's results; the only difference now is the presence, as a factor in T_{if}, of the "form factor" $(2\pi)^{\frac{3}{2}} F_i(\underset{\sim}{k}' - \underset{\sim}{\xi}\underset{\sim}{k})$ instead of its zero-range counterpart, $-4\pi (2\alpha)^{\frac{1}{2}} \hbar^2/2M_{AB}$. The integral in eq. (7) may be carried out for L = 0 final states which are linear combinations of $r^n \exp(-\beta r)/r$; the basic integral is [1,3,8]

$$\int \psi^{(-)}_{\underset{\sim}{k}'}(\underset{\sim}{\xi}'\underset{\sim}{r})^* \frac{e^{-\beta r}}{r} \psi^{(+)}_{\underset{\sim}{k}}(\underset{\sim}{r})\, d\underset{\sim}{r} = C_0(\eta) C_0(\eta') \frac{4\pi}{Q^2+\beta^2} \times$$

$$\left[\frac{K'^2 - (k+i\beta)^2}{Q^2+\beta^2}\right]^{i\eta} \left(\frac{k+K'+i\beta}{k-K'+i\beta}\right)^{i\eta'} F(1+i\eta, -i\eta'; 1; z), \qquad (8)$$

$$z = 1 - \frac{(K'-k)^2+\beta^2}{Q^2+\beta^2}, \qquad (9)$$

where $\underset{\sim}{Q} = \underset{\sim}{K}' - \underset{\sim}{k}$, $\underset{\sim}{K}' = \underset{\sim}{\xi}'\underset{\sim}{k}'$, $C_0(\eta) = \Gamma(1+i\eta)\exp(-\pi\eta/2)$, and η and η' are the charge parameters in the initial and final state respectively. Note that $0 \leq z < 1$ for all angles of scattering.

For final states with $L \neq 0$ the calculation is more difficult, because of the presence of $Y_{LM}(\hat{r})$ in ϕ_f. However, it is always possible to find a differential operator $\Omega_{LM}(\underset{\sim}{\sigma})$ such that

$$r^L Y_{LM}(\hat{r}) = \lim_{\underset{\sim}{\sigma}\to 0} \Omega_{LM}(\underset{\sim}{\sigma})\, e^{i\underset{\sim}{\sigma}\cdot\underset{\sim}{r}} \qquad (10)$$

thus eq. (7) can be evaluated analytically for final states of the form

$$r^L Y_{LM}(\hat{r})\, e^{-\beta r}/r. \qquad (11)$$

In this case the basic integral is

$$\int \underset{\sim}{\psi}^{(-)}_{\underset{\sim}{k}'}(\underset{\sim}{\xi}'\underset{\sim}{r})^* \frac{e^{i\underset{\sim}{\sigma}\cdot\underset{\sim}{r} - \underset{\sim}{\beta} r}}{r} \underset{\sim}{\psi}^{(+)}_{\underset{\sim}{k}}(\underset{\sim}{r})\, d\underset{\sim}{r}\,, \tag{12}$$

which may be obtained from Nordsieck's paper.[8]

Unfortunately, in the important asymptotic region where the Coulomb wave functions are large, (11) is not a very good approximation to $\underset{\sim}{\phi}_f$ (because of the factor r^L). At present there is no completely satisfactory way of avoiding this difficulty. However, if $\underset{\sim}{\phi}_i$ is an L = 0 state, we may obtain an expression in closed form by removing $\underset{\sim}{\phi}_f(\underset{\sim}{P}')$ instead of $F_i(\underset{\sim}{P})$ from under the integral sign in eq. (2):

$$T_{if} \cong (2\pi)^{\frac{3}{2}} \underset{\sim}{\phi}_f(\underset{\sim}{k} - \underset{\sim}{\xi}'\underset{\sim}{k}') \int \underset{\sim}{\psi}^{(-)}_{\underset{\sim}{k}'}(\underset{\sim}{R})^* V_{AB}(\underset{\sim}{R})\, \underset{\sim}{\phi}_i(\underset{\sim}{R})\, \underset{\sim}{\psi}^{(+)}_{\underset{\sim}{k}}(\underset{\sim}{\xi}\underset{\sim}{R})\, d\underset{\sim}{R}\,. \tag{13}$$

Now, since $V_{AB}\,\underset{\sim}{\phi}_i$ may be written in terms of $\underset{\sim}{\phi}_i$ and $\nabla^2\underset{\sim}{\phi}_i$, an approximation to $\underset{\sim}{\phi}_i$ with terms of the form $r^n\exp(-\underset{\sim}{\beta} r)/r$ will lead to integrals which can be evaluated by means of eq. (8).

Finally, we should like to mention that at very large momentum transfers the presence of the finite-range form factor $F_i(\underset{\sim}{k}' - \underset{\sim}{\xi}\underset{\sim}{k})$ may reduce the cross section drastically.[9] In this case the "peaking approximation" used above is of dubious value, since $F_i(\underset{\sim}{P})$ is rapidly varying in the vicinity of the scattering transform peak. However, such large momentum transfers indicate that the reaction is taking place within or near the nuclear surface, and under these conditions the Coulomb wave approximation is not expected to be valid in any event.

Footnote

† This "peaking approximation" for Coulomb waves has also been used by other authors; see, for example, refs. [6,7].

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